Math 132 : Differential Topology

§ Lefschetz fixed point theorem

Let M be a compact, oriented manifold, and $f : M \to M$ a smooth map.

The global Lefschetz number of f, denoted $L(f)$, is

$$L(f) = I(\Delta, \mathrm{graph}(f)).$$

The followings are immediate consequences of the intersection theory:

Prop $L(f)$ is a homotopy invariant.

Thm (smooth Lefschetz fixed point theorem)

If $L(f) \neq 0$, then f has a fixed point.

fixed points of f $\Updownarrow$ intersection points $\Delta \cap \mathrm{graph}(f)$

Prop If f is homotopic to the identity, then $L(f) = \chi(M)$.

In particular, if M admits a smooth map $f : M \to M$ homotopic to the identity that has no fixed points, then $\chi(M) = 0$.

Ex Rotation of S^1 by an angle $\theta \neq 0 \mod 2\pi$ has no fixed points, so $\chi(S^1) = 0$.

We say a map $f : M \to M$ is Lefschetz if $\Delta \pitchfork \mathrm{graph}(f)$.

Clearly,

Prop Every map $f : M \to M$ is homotopic to a Lefschetz map.

Note, $\Delta \pitchfork \mathrm{graph}(f)$ at (x,x) $\Leftrightarrow \Delta_x + \mathrm{graph}(df_x) = T_xM \times T_xM$

$\Leftrightarrow \Delta_x \cap \mathrm{graph}(df_x) = 0$

$\Leftrightarrow df_x$ has no eigenvector of eigenvalue 1

(i.e. $\det(df_x - I) \neq 0$)

We say a fixed point x of f is a Lefschetz fixed point if holds.

For a Lefschetz fixed point x of f, the local Lefschetz number of f at x, $L_x(f)$, is the orientation number $\in \{\pm 1\}$ of $(x,x) \in \Delta \cap \mathrm{graph}(f)$.

Clearly, for Lefschetz maps,

$$L(f) = \sum_{f(x)=x} L_x(f).$$

sum over fixed points.

Prop $L_x(f) = \mathrm{sign}(\det(df_x - I))$.

proof) The proof is a simple linear algebra: if $\{v_1, \cdots, v_m\}$ is a positively oriented ordered basis for T_xM, then

$$\begin{aligned}
L_x(f) &= \mathrm{sign}\{\underbrace{(v_1,v_1), \cdots, (v_m,v_m)}_{\text{positive for } T_{(x,x)}\Delta}, \underbrace{(v_1, df_x(v_1)), \cdots, (v_m, df_x(v_m))}_{\text{positive for } T_{(x,x)}\mathrm{graph}(f)}\} \\
&= \mathrm{sign}\{(v_1,v_1), \cdots, (v_m,v_m), (0,(df_x - I)v_1), \cdots, (0,(df_x - I)v_m)\} \\
&= \mathrm{sign}\{(v_1,0), \cdots, (v_m,0), (0,(df_x - I)v_1), \cdots, (0,(df_x - I)v_m)\} \\
&= \mathrm{sign}(\det(df_x - I)). \blacksquare
\end{aligned}$$

Ex (local pictures in 2d)

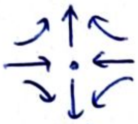
Assume that $df_x = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ with $\alpha_1, \alpha_2 > 0$.

Case 1: If either $\alpha_1, \alpha_2 > 1$ or $\alpha_1, \alpha_2 < 1$, then $L_x(f) = +1$.


source


sink

Case 2: If $\alpha_1 < 1 < \alpha_2$, then $L_x(f) = -1$

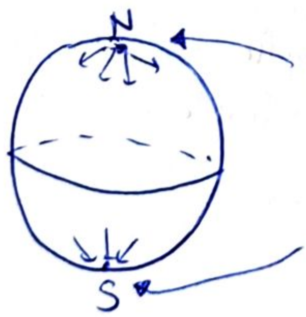

saddle

Ex (Euler characteristic from local Lefschetz numbers)

Consider the map $f: S^2 \to S^2$ given by a "downward flow"

$$f_t(x) = \pi\left(x + (0, 0, -\tfrac{t}{2})\right), \quad 0 < t < 2,$$

where $\pi: \mathbb{R}^3 \setminus \{0\} \to S^2$ is the projection.



source at the north pole

sink at the south pole

$\rightsquigarrow \chi(S^2) = L(f_1) = 1 + 1 = 2.$

Cor Every map $f: S^2 \to S^2$ homotopic to the identity must have a fixed point. In particular, the antipodal map $x \mapsto -x$ is not homotopic to the identity.

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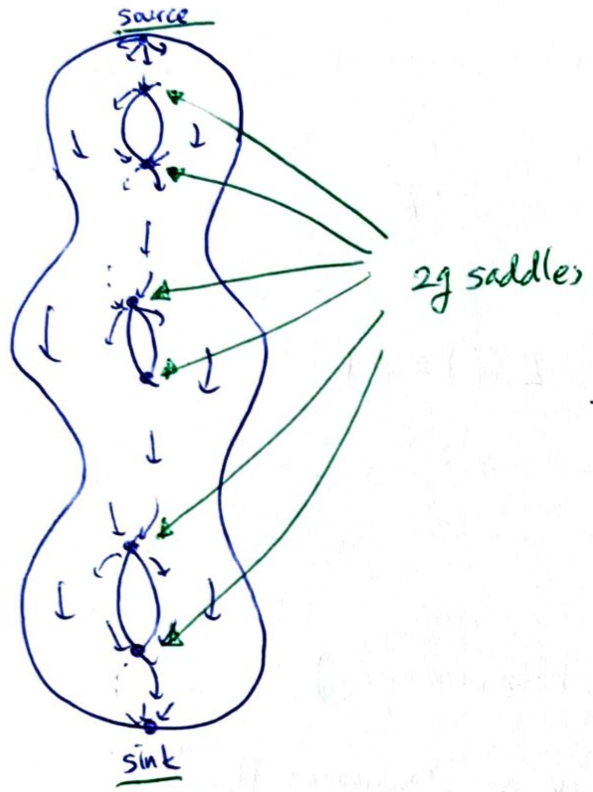
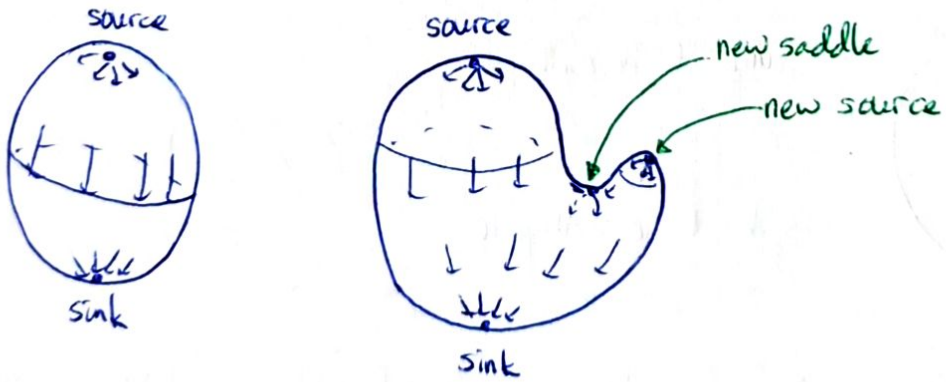
More generally, a genus g surface admits a Lefschetz map homotopic to the identity that has 1 source, 1 sink, and $2g$ saddles:

Σ_g



$$\rightsquigarrow \chi(\Sigma_g) = 2-2g$$

It is also instructive to see how the fixed points get created and annihilated in pairs when we deform the "height function":



$$\chi(S^2) = 1+1 \quad = \quad 1+1+(1-1)$$